Real hypersurfaces in the complex hyperbolic quadric with Reeb parallel shape operator

Young Jin Suh & Doo Hyun Hwang

Annali di Matematica Pura ed Applicata (1923 -)

ISSN 0373-3114
Volume 196
Number 4

Annali di Matematica (2017)
196:1307-1326
DOI 10.1007/s10231-016-0617-0

Annali di Matematica (2017) 196:1307–1326
DOI 10.1007/s10231-016-0617-0

Real hypersurfaces in the complex hyperbolic quadric with Reeb parallel shape operator

Young Jin Suh[1] · **Doo Hyun Hwang**[1]

Received: 14 May 2016 / Accepted: 7 October 2016 / Published online: 21 October 2016

Abstract First, we introduce the notion of shape operator of Codazzi type for real hypersurfaces in the complex quadric $Q^{m*} = SO^o_{m,2}/SO_m SO_2$. Next, we give a complete proof of non-existence of real hypersurfaces in $Q^{m*} = SO^o_{m,2}/SO_m SO_2$ with shape operator of Codazzi type. Motivated by this result, we give a complete classification of real hypersurfaces in $Q^{m*} = SO^o_{m,2}/SO_m SO_2$ with Reeb parallel shape operator.

Keywords Reeb parallel hypersurface · Kähler structure · Complex conjugation · Complex quadric

Mathematics Subject Classification Primary 53C40; Secondary 53C55

1 Introduction

As examples of some Hermitian symmetric spaces of rank 2, usually we can consider the Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, respectively (see [11,12] and [15–18]). These are Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$ on $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$. There are exactly two types of singular tangent vector fields X on $SU_{m+2}/S(U_2U_m)$

This work was supported by Grant Proj. No. NRF-2015-R1A2A1A-01002459 from National Research Foundation of Korea.

✉ Young Jin Suh
yjsuh@knu.ac.kr

Doo Hyun Hwang
engus0322@knu.ac.kr

1 Department of Mathematics, College of Natural Sciences, Kyungpook National University, Daegu 41566, Republic of Korea

and $SU_{2,m}/S(U_2U_m)$, which are characterized by the geometric properties $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

As another example of Hermitian symmetric space of compact type with rank 2 different from the above ones, we can consider a complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Reckziegel [13], Smyth [14] and Suh [19,20], and [21]). The complex quadric can also be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [6]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple, (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [5] and Reckziegel [13]).

About the latter part of twentieth century, many geometers have investigated some real hypersurfaces in Hermitian symmetric spaces of rank 1 like the complex projective space $\mathbb{C}P^m$ or the complex hyperbolic space $\mathbb{C}H^m$. In the complex projective space $\mathbb{C}P^m$ and the quaternionic projective space $\mathbb{H}P^m$, a characterization with isometric Reeb flow was obtained by Okumura [8], $\mathcal{D}$-parallel shape operator $\nabla_{\mathcal{D}}A = 0$ by Pérez [9], and $\mathcal{D}$-parallel curvature tensor $\nabla_{\mathcal{D}}R = 0$ by Pérez and Suh [10], respectively, where $\mathcal{D} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, $\xi_i = -J_iN$, $i = 1, 2, 3$.

Now, let us introduce complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_2SO_m$, which can be regarded as a Hermitian symmetric space with rank 2 of noncompact type. Here, we consider a real hypersurface M in Q^{m*} with Reeb parallel shape operator, that is, $\nabla_\xi S = 0$ for the shape operator S of M in Q^{m*} along the Reeb direction $\xi = -JN$ on M, where J denotes the Kähler structure on Q^{m*}.

In order to give a complete classification of real hypersurfaces in Q^{m*} with Reeb parallel shape operator, first we will consider a problem of non-existence for real hypersurfaces in Q^{m*} with parallel shape operator. More generally, we consider the shape operator S of M in Q^{m*} satisfying $(\nabla_X S)Y = (\nabla_Y S)X$ for any vector fields X and Y on M in Q^{m*}. In this case, the shape operator is said to be of *Codazzi type*. In this paper, we want to give a property of non-existence for real hypersurfaces in the complex hyperbolic quadric Q^{m*} whose shape operator is of Codazzi type as follows:

Theorem 1.1 *There do not exist any real hypersurfaces in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with shape operator of Codazzi type.*

Next we will consider parallel shape operator for M in Q^{m*}. Since the parallel shape operator S naturally satisfy the condition of Codazzi type, we can also assert the following

Corollary 1.2 *There do not exist any real hypersurfaces in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with parallel shape operator.*

Apart from the complex structure J, there is another distinguished geometric structure on Q^{m*}. Namely, a vector subbundle $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^{m*}$, which consists of all complex conjugations defined on the complex hyperbolic quadric Q^{m*}. The vector bundle $\mathfrak{A}$ contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*} and becomes a parallel rank 2-subbundle of End TQ^{m*}. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*}.

Recall that a nonzero tangent vector $W \in T_zQ^{m*}$ is called *singular* if it is tangent to more than one maximal flat in Q^{m*}. Here maximal flat means a 2-dimensional curvature flat totally geodesic submanifold in Q^{m*}. Such a maximal flat always exists, because the rank of Q^{m*} is 2. There are two types of singular tangent vectors for the complex quadric Q^{m*} as follows:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic,

where $V(A) = \{X \in T_{[z]}Q^{m*} | AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^{m*} | AX = -X\}$, $[z] \in Q^{m*}$, respectively denote the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$.

Here we note that the unit normal N is said to be $\mathfrak{A}$-principal if N is invariant under the complex conjugation A, that is, $AN = N$.

For the complex hyperbolic space $\mathbb{C}H^m$, a full classification of real hypersurfaces with isometric Reeb flow was obtained by Montiel and Romero in [7]. They proved that the Reeb flow on a real hypersurface in $\mathbb{C}H^m = SU_{m,1}/S(U_m U_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^m$ for some $k \in \{0, \ldots, m-1\}$ or horospheres.

The classification problem for real hypersurfaces with isometric Reeb flow for the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ and the complex hyperbolic 2-plane Grassmannian $G_2^*(\mathbb{C}^{m+2})$ were solved by Suh [15,18] and Suh [17], respectively. The Reeb flow on a real hypersurface in $G_2^*(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2^*(\mathbb{C}^{m+1}) \subset G_2^*(\mathbb{C}^{m+2})$.

Now let us consider such a situation in the complex hyperbolic quadric Q^{m*}. In view of the previous two results, a natural expectation might be that the classification involves at least the totally geodesic $Q^{m-1*} \subset Q^{m*}$. But in the paper due to Suh [22], we investigate this problem for the complex hyperbolic quadric $Q^{m*} = SO_{m,2}/SO_m SO_2$ as follows:

Theorem A *Let M be a real hypersurface in the complex hyperbolic quadric $Q^{m*} = SO^o_{2,m}/SO_m SO_2$, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

When the shape operator S of M in Q^{m*} is Reeb parallel, that is, $\nabla_\xi S = 0$ along the direction of the structure vector field $\xi = -JN$, we say that the shape operator is *Reeb parallel*. Moreover, we say that the Reeb principal curvature is constant if the function α defined by $\alpha = g(S\xi, \xi)$ is constant. Motivated by these results, we give a complete classification of real hypersurfaces in the complex hyperbolic quadric Q^{m*} with Reeb parallel shape operator as follows:

Theorem 1.3 *Let M be a Hopf real hypersurface in complex hyperbolic quadric Q^{m*}, $m \geq 3$, with Reeb parallel shape operator and non-vanishing Reeb curvature. Then M is an open part of the following:*

(1) a tube around the totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$, where $m = 2k$,
(2) a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular,
(3) a tube around the totally geodesic Hermitian symmetric space Q^{m-1*} embedded in Q^{m*},
(4) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},

(5) a tube around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form,
or otherwise
(6) M has two distinct constant prinicipal curvatures given by

$$\alpha, \quad \lambda = \frac{\alpha^2 - 2}{\alpha}$$

with multiplicities m and $m-1$, respectively.

2 The complex hyperbolic quadric

Let us denote by $\mathbb{C}_1^{m+2}$ an indefinite complex Euclidean space $\mathbb{C}^{m+2}$, on which the indefinite Hermitian product

$$H(z, w) = -z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_{m+2}\bar{w}_{m+2}$$

can be considered. The scalar product defined by the real part of $H(z, w)$ is an indefinite Riemannian metric of index 2 on $\mathbb{C}^{m+2}$. Here the complex hyperbolic space $\mathbb{C}H^{m+1}$ is the set of all complex 1-dimensional subspaces in $\mathbb{C}_1^{m+2}$, on which the indefinite Hermitian poduct $H(z, w)$ is negative definite.

The homogeneous quadratic equation $z_1^2 + \cdots + z_m^2 - z_{m+1}^2 - z_{m+2}^2 = 0$ defines a noncompact complex hyperbolic quadric $Q^{m*} = SO_{2,m}^o/SO_2SO_m$ which can be immersed in the $(m+1)$-dimensional complex hyperbolic space $\mathbb{C}H^{m+1} = SU_{1,m+1}/S(U_{m+1}U_1)$. The complex hypersurface Q^{m*} in $\mathbb{C}H^{m+1}$ is known as the m-dimensional complex hyperbolic quadric. The complex structure J on $\mathbb{C}H^{m+1}$ naturally induces a complex structure on Q^{*m} which we will denote by J as well. We equip Q^{m*} with the Riemannian metric g which is induced from the Bergmann metric on $\mathbb{C}H^{m+1}$ with constant holomorphic sectional curvature -4. For $m \geq 2$ the triple, (Q^{m*}, J, g) is a Hermitian symmetric space of rank two, and its minimal sectional curvature is equal to -4. The 1-dimensional quadric Q^{1*} is isometric to the 2-dimensional real hyperbolic space $\mathbb{R}H^2 = SO_{1,2}^o/SO_1SO_2$. The 2-dimensional complex quadric Q^{2*} is isometric to the Riemannian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$. We will assume $m \geq 3$ for the main part of this paper.

For a nonzero vector $z \in \mathbb{C}_1^{m+2}$, we denote by $[z]$ the complex span of z, that is, $[z] = \{\lambda z \mid \lambda \in \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}H^{m+1}$. As usual, for each $[z] \in \mathbb{C}H^{m+1}$ we identify $T_{[z]}\mathbb{C}H^{m+1}$ with the orthogonal complement $\mathbb{C}_1^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}_1^{m+2}$. For $[z] \in Q^{m*}$ the tangent space $T_{[z]}Q^{m*}$ can then be identified canonically with the orthogonal complement $\mathbb{C}_1^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}_1^{m+2}$. Note that $\bar{z} \in \nu_{[z]}Q^{m*}$ is a unit normal vector of Q^{m*} in $\mathbb{C}H^{m+1}$ at the point $[z]$.

We denote by $A_{\bar{z}}$ the shape operator of Q^{m*} in $\mathbb{C}H^{m+1}$ with respect to $\bar{z}$. Then we have $A_{\bar{z}}w = \overline{w}$ for all $w \in T_{[z]}Q^{m*}$, that is, $A_{\bar{z}}$ is just complex conjugation restricted to $T_{[z]}Q^{m*}$. The shape operator $A_{\bar{z}}$ is an antilinear involution on the complex vector space $T_{[z]}Q^{*m}$ and

$$T_{[z]}Q^{m*} = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}_1^{m+2} \cap T_{[z]}Q^{m*}$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}_1^{m+2} \cap T_{[z]}Q^{m*}$ is the (-1)-eigenspace of $A_{\bar{z}}$. Geometrically, this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^{m*}$. Recall that a real structure on a complex vector space V is by definition an antilinear involution $A : V \to V$. Since the normal space $\nu_{[z]}Q^{m*}$ of Q^{m*} in $\mathbb{C}H_1^{m+1}$ at $[z]$ is a complex subspace of $T_{[z]}\mathbb{C}H^{m+1}$ of complex dimension one, every normal vector in $\nu_{[z]}Q^{m*}$ can be written as $\lambda\bar{z}$ with some $\lambda \in \mathbb{C}$. The shape operators $A_{\lambda\bar{z}}$ of Q^{m*} define a rank two vector subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$.

The derivative of the second fundamental form of the embedding $Q^{m*} \subset \mathbb{C}H^{m+1}$ is given by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on Q^{m*}, where $\bar{\nabla}$ and q denote the Levi-Civita connection and a certain 1-form on TQ^{m*}, respectively. So the set of all complex conjugations $\mathfrak{A}$ becomes a parallel subbundle of $\mathrm{End}(TQ^{m*})$. For $\lambda \in S^1 \subset \mathbb{C}$, we again get a real structure $A_{\lambda\bar{z}}$ on $T_{[z]}Q^{m*}$. Because, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}}w &= A_{\lambda\bar{z}}A_{\lambda\bar{z}}w = A_{\lambda\bar{z}}\lambda\bar{w} \\ &= \lambda A_{\bar{z}}\lambda\bar{w} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. We thus have an S^1-subbundle of $\mathfrak{A}$ consisting of real structures on the tangent spaces of Q^{m*}.

The Gauss equation for the complex hypersurface $Q^{m*} \subset \mathbb{C}H^{m+1}$ implies that the Riemannian curvature tensor R of Q^{m*} can be expressed in terms of the Riemannian metric g, the complex structure J and a generic real structure A in $\mathfrak{A}$:

$$\begin{aligned} R(X,Y)Z = &-g(Y,Z)X + g(X,Z)Y \\ &- g(JY,Z)JX + g(JX,Z)JY + 2g(JX,Y)JZ \\ &- g(AY,Z)AX + g(AX,Z)AY \\ &- g(JAY,Z)JAX + g(JAX,Z)JAY. \end{aligned}$$

Note that the complex structure J anti-commutes with each endomorphism $A \in \mathfrak{A}$, that is, $AJ = -JA$.

Basic complex linear algebra shows that for every unit tangent vector $W \in T_{[z]}Q^{m*}$ there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$.

3 The maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of TM

Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M. If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface (See Dragomir and Perrone [4]). The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathcal{F}$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$. We denote by νM the normal bundle of M.

We first introduce some notations. For a fixed real structure $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$, we decompose AX into its tangential and normal component, that is,

$$AX = BX + \rho(X)N,$$

where BX is the tangential component of AX and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

Since $JX = \phi X + \eta(X)N$ and $A\xi = B\xi + \rho(\xi)N$, we also have

$$\rho(X) = g(\phi X, B\xi) + \eta(X)\rho(\xi) = \eta(B\phi X) + \eta(X)\rho(\xi).$$

We also define

$$\delta = g(N, AN) = g(JN, JAN) = -g(JN, AJN) = -g(\xi, A\xi).$$

At each point $[z] \in M$ we define

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\},$$

which is the maximal $\mathfrak{A}_{[z]}$-invariant subspace of $T_{[z]}M$. Then by using the same method for real hypersurfaces in complex hyperbolic quadric Q^{m*} as in Berndt and Suh [2], we get the following

Lemma 3.1 *Let M be a real hypersurface in complex hyperbolic quadric Q^{m*}. Then the following statements are equivalent:*

(i) *The normal vector $N_{[z]}$ of M is $\mathfrak{A}$-principal,*
(ii) $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$,
(iii) *There exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $AN_{[z]} \in \mathbb{C}\nu_{[z]}M$.*

Assume now that the normal vector $N_{[z]}$ of M is not $\mathfrak{A}$-principal. Then there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 < t \leq \frac{\pi}{4}$. This implies

$$\begin{aligned} N_{[z]} &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi_{[z]} &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi_{[z]} &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

and therefore, $\mathcal{Q}_{[z]} = T_{[z]}Q^m \ominus ([Z_1] \oplus [Z_2])$ is strictly contained in $\mathcal{C}_{[z]}$. Moreover, we have

$$A\xi_{[z]} = B\xi_{[z]} \quad \text{and} \quad \rho(\xi_{[z]}) = 0.$$

We have

$$\begin{aligned} g(B\xi_{[z]} + \delta\xi_{[z]}, N_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, \xi_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, B\xi_{[z]} + \delta\xi_{[z]}) &= \sin^2(2t), \end{aligned}$$

where the function δ denotes $\delta = -g(\xi, A\xi) = -(\sin^2 t - \cos^2 t) = \cos 2t$. Therefore,

$$U_{[z]} = \frac{1}{\sin(2t)}(B\xi_{[z]} + \delta\xi_{[z]})$$

is a unit vector in $\mathcal{C}_{[z]}$ and

$$\mathcal{C}_{[z]} = \mathcal{Q}_{[z]} \oplus [U_{[z]}] \text{ (orthogonal direct sum).}$$

If $N_{[z]}$ is not $\mathfrak{A}$-principal at $[z]$, then N is not $\mathfrak{A}$-principal in an open neighborhood of $[z]$, and therefore, U is a well-defined unit vector field on that open neighborhood. We summarize this in the following

Lemma 3.2 *Let M be a real hypersurface in complex hyperbolic quadric Q^{m*} whose unit normal $N_{[z]}$ is not $\mathfrak{A}$-principal at $[z]$. Then there exists an open neighborhood of $[z]$ in M and a section A in $\mathfrak{A}$ on that neighborhood consisting of real structures such that*

(i) *$A\xi = B\xi$ and $\rho(\xi) = 0$,*
(ii) *$U = (B\xi + \delta\xi)/||B\xi + \delta\xi||$ is a unit vector field tangent to $\mathcal{C}$*
(iii) *$\mathcal{C} = \mathcal{Q} \oplus [U]$.*

4 Tubes around the totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ and horospheres

We assume that m is even, say $m = 2k$. The map

$$\mathbb{C}H^k \to Q^{2k^*} = SO^o_{2,2k}/SO_2SO_{2k} \subset \mathbb{C}H^{2k+1},$$

is defined by $[z_1, \ldots, z_{k+1}] \mapsto [z_1, \ldots, z_{k+1}, iz_1, \ldots, iz_{k+1}]$, provides an embedding of $\mathbb{C}H^k$ into Q^{2k^*} as a totally geodesic complex submanifold in $\mathbb{C}H^{2k+1}$, where

$$Q^{*2k} = \{[z_1, \cdots, z_{2k+2}] \in \mathbb{C}H^{2k+1} | -z_1^2 + z_2^2 + \cdots z_{k+1}^2 - z_{k+2}^2 + z_{k+3}^2 + \cdots + z_{2k+2}^2 = 0\}$$

can be regarded as the set of negative 2-planes in indefinite Euclidean space $\mathbb{R}^{2k+2}_2$, that is, a real hyperbolic Grassmannian manifold. Of course, it can be easily checked that the point $[z_1, \ldots, z_{k+1}, iz_1, \ldots, iz_{k+1}]$ belongs to Q^{2k^*}.

Consider the standard embedding of $U_{1,k}$ into $SO^o_{2,2k}$ which is determined by the Lie algebra embedding in such a way that

$$\mathfrak{u}_{1,k} \to \mathfrak{so}_{2,2k}, \quad C + Di \to \begin{pmatrix} C & -D \\ D & C \end{pmatrix},$$

where $C, D \in M_{k+1,k+1}(\mathbb{R})$ which satisfy, respectively, ${}^tCgC = g$ and ${}^tDgD = g$ for the signature $(1, k)$ of the indefinite Riemannian metric g on $\mathbb{R}^{k+1}_1$ defined by $g(X, Y) = -x_1y_1 + x_2y_2 + \cdots + x_{k+1}y_{k+1}$ for any $X, Y \in \mathbb{R}^{k+1}$.

We define a complex structure j on $\mathbb{C}^{2k+2}_1$ by

$$j(z_1, \ldots, z_{k+1}, z_{k+2}, \ldots, z_{2k+2}) = (-z_{k+2}, \ldots, -z_{2k+2}, z_1, \ldots, z_{k+1}).$$

Note that $ij = ji$. We can then identify $\mathbb{C}^{2k+2}_1$ with $\mathbb{C}^{k+1}_1 \oplus j\mathbb{C}^{k+1}$ and get

$$T_{[z]}\mathbb{C}H^k = \{X + jiX \mid X \in \mathbb{C}^{k+1}_1 \ominus [z]\} = \{X + ijX \mid X \in V(A_{\bar{z}})\}.$$

Note that the complex structure j on $\mathbb{C}^{2k+2}_1$ corresponds to the complex structure J on $T_{[z]}Q^{2k^*}$ via the obvious identifications. For the normal space $\nu_{[z]}\mathbb{C}H^k$ of $\mathbb{C}H^k$ at $[z]$, we have

$$\nu_{[z]}\mathbb{C}H^k = A_{\bar{z}}(T_{[z]}\mathbb{C}H^k) = \{X - ijX \mid X \in V(A_{\bar{z}})\}.$$

It is easy to see that both the tangent bundle and the normal bundle of $\mathbb{C}H^k$ consist of $\mathfrak{A}$-isotropic singular tangent vectors of Q^{2k*}.

We will now calculate the principal curvatures and principal curvature spaces of the tube around $\mathbb{C}H^k$ in Q^{2k*}. Let N be a unit normal vector of $\mathbb{C}H^k$ in Q^{2k*} at $[z] \in \mathbb{C}H^k$. Since by Theorem A, the unit normal N is $\mathfrak{A}$-isotropic. Then the four vectors N, JN, AN and JAN are pairwise orthonormal and the normal Jacobi operator $\bar{R}_N$ is given by

$$\begin{aligned}\bar{R}_N Z = \bar{R}(Z, N)N &= -Z + g(Z, N)N - 3g(Z, JN)JN \\ &\quad + g(Z, AN)AN + g(Z, JAN)JAN.\end{aligned}$$

From this, by using that N is $\mathfrak{A}$-isotropic, $\bar{R}_N N = \bar{R}(N, N)N = 0$, $\bar{R}_N AN = \bar{R}(AN, N)N = 0$, $\bar{R}_N JAN = 0$, and $\bar{R}_N JN = -4JN$. This implies readily that $\bar{R}_N$ has the three eigenvalues $0, -1$ and -4 with corresponding eigenspaces $\mathbb{R}N \oplus [AN]$, $T_{[z]}Q^{2k*} \ominus ([N] \oplus [AN])$ and $\mathbb{R}JN$. Since $[N] \subset \nu_{[z]}\mathbb{C}H^k$ and $[AN] \subset T_{[z]}\mathbb{C}H^k$, we conclude that both $T_{[z]}\mathbb{C}H^k$ and $\nu_{[z]}\mathbb{C}H^k$ are invariant under $\bar{R}_N$.

To calculate the principal curvatures of the tube around $\mathbb{C}H^k$ we use the Jacobi field method. Let γ be the geodesic in Q^{2k*} with $\gamma(0) = [z]$ and $\dot{\gamma}(0) = N$ and denote by $\gamma^{\perp}$ the parallel subbundle of TQ^{2k*} along γ defined by $\gamma^{\perp}_{\gamma(t)} = T_{[\gamma(t)]}Q^{2k*} \ominus \mathbb{R}\dot{\gamma}(t)$. Moreover, define the $\gamma^{\perp}$-valued tensor field $R^{\perp}_{\gamma}$ along γ by $R^{\perp}_{\gamma(t)}X = R(X, \dot{\gamma}(t))\dot{\gamma}(t)$. Now consider the $\mathrm{End}(\gamma^{\perp})$-valued differential equation

$$Y'' + R^{\perp}_{\gamma} \circ Y = 0.$$

Let D be the unique solution of this differential equation with initial values

$$D(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where the decomposition of the matrices is with respect to

$$\gamma^{\perp}_{[z]} = T_{[z]}\mathbb{C}H^k \oplus (\nu_{[z]}\mathbb{C}H^k \ominus \mathbb{R}N)$$

and I denotes the identity transformation on the corresponding space. Then the shape operator $S(r)$ of the tube around $\mathbb{C}H^k$ with respect to $-\dot{\gamma}(r)$ is given by

$$S(r) = D'(r) \circ D^{-1}(r).$$

If we decompose $\gamma^{\perp}_{[z]}$ further into

$$\gamma^{\perp}_{[z]} = (T_{[z]}\mathbb{C}H^k \ominus [AN]) \oplus [AN] \oplus (\nu_{[z]}\mathbb{C}H^k \ominus [N]) \oplus \mathbb{R}JN,$$

we get by explicit computation that

$$S(r) = \begin{pmatrix} \tanh(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \coth(r) & 0 \\ 0 & 0 & 0 & 2\coth(2r) \end{pmatrix}$$

with respect to that decomposition. Here let us check that $SJN = 2\coth(2r)JN$ for $M \subset Q^{2k*}$. Since $\bar{R}_N JN = -4JN$, we have $Y'' - 4Y = 0$ for a geodesic γ such that

$\gamma(0) = [z]$ and $\dot{\gamma}(0) = N$. The solution vector field $Y(r)$ of the Jacobi equation becomes

$$Y(r) = (c_1 \cosh(2r) + c_2 \sinh(2r))E_X(r),$$

where the initial condition is given by $0 = Y(0) = c_1 E_X(0) = c_1 X$ and $X = Y'(0) = 2c_2 E_X(0) = 2c_2 X$ and the vector field $E_X(r)$ is defined by the parallel displacement of the vector $E_X(0) = X$ along the curve γ.

Here we know that the solution vector field can be obtained by $Y(r) = D(r)E_X(r) = \frac{1}{2}\sinh(2r)E_X(r)$. From this, together with the definition of the shape operator, it follows that

$$\begin{aligned}\frac{1}{2}\sinh(2r)S(r)E_X(r) = S(r)Y(r) &= D'(r)D^{-1}(r)Y\\ &= D'(r)E_X(r) = \cosh(2r)E_X(r).\end{aligned}$$

This implies that $S(r)E_X(r) = 2\coth(2r)E_X(r)$, which means $S(r)JN = 2\coth(2r)JN$. By using the similar method we can calculate the other principal curvatures. Therefore, the tube around $\mathbb{C}H^k$ has four distinct constant principal curvatures $\tanh(r)$, 0, $\coth(r)$ and $2\coth(2r)$ (unless $m = 2$ in which case there are only two distinct constant principal curvatures 0 and $2\coth(2r)$). The corresponding principal curvature spaces are $T_{[z]}\mathbb{C}H^k \ominus [AN]$, $[AN]$, $\nu_{[z]}\mathbb{C}H^k \ominus [N]$ and $\mathbb{R}JN$, respectively, where we identify the subspaces obtained by parallel translation along γ from $[z]$ to $\gamma(r)$. This shows that the tube is a *Hopf* hypersurface.

Note that the parallel translate of $[AN]$ corresponds to $\mathcal{C} \ominus \mathcal{Q}$, the parallel translate of $[N]$ corresponds to $\mathbb{C}\nu M$, and the parallel translate of $\mathbb{R}JN$ corresponds to $\mathcal{F}$. Moreover, we have $A(T_{[z]}\mathbb{C}H^k \ominus [AN]) = \nu_{[z]}\mathbb{C}H^k \ominus [N]$.

When M becomes an open part of a horosphere in Q^{2k^*} whose center at infinity in the equivalence class of an $\mathfrak{A}$-isotropic geodesic in Q^{2k^*}, by using the results in Suh [22] and taking the limit to the above principal curvatures as $r \to \infty$, we can calculate that it has three distinct constant prinicipal curvatures 1, 0, 1 and 2 corresponding to the same principal curvature spaces mentioned above.

Since JN is a principal curvature vector, we conclude that every tube around $\mathbb{C}H^k$ is a Hopf hypersurface. We also see that all principal curvature spaces orthogonal to $\mathbb{R}JN$ are J-invariant. Thus, if ϕ denotes the structure tensor field on the tube which is induced by J, we get $S\phi = \phi S$. Since the Kähler structure on Q^{m*} is parallel, we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = g((S\phi - \phi S)X, Y)$$

for all $X, Y \in TM$. As ξ is a Killing vector field if and only if $\nabla\xi$ is a skew-symmetric tensor field, we see that the Reeb flow on M is isometric if and only if $S\phi = \phi S$.

We summarize the previous discussion in the following proposition.

Proposition 4.1 *Let M be the tube around the totally geodesic $\mathbb{C}H^k$ in the complex hyperbolic quadric Q^{2k^*}, $k \geq 2$, or the horosphere in Q^{2k^*} whose center at infinity is in the equivalence class of an $\mathfrak{A}$-isotropic singular geodesic in Q^{2k^*}. Then the following statements hold:*

(i) *M is a Hopf hypersurface,*
(ii) *The tangent bundle TM and the normal bundle νM of M consist of $\mathfrak{A}$-isotropic singular tangent vectors of Q^{2k^*},*
(iii) *M has four(or three) distinct constant principal curvatures. Their values and corresponding principal curvature spaces and multiplicities are given in the following Table 1. The real structure A determined by the $\mathfrak{A}$-isotropic unit normal vector at $[z]$ maps $T_{[z]}\mathbb{C}H^k \ominus (\mathcal{C}_{[z]} \ominus \mathcal{Q}_{[z]})$ onto $\nu_{[z]}\mathbb{C}H^k \ominus \mathbb{C}\nu_{[z]}M$, and vice versa,*

Table 1 Principal curvatures of M

Principal curvature	Eigenspace	Multiplicity
$2\coth(2r)$, 2	$\mathcal{F}$	1
0	$\mathcal{C} \ominus \mathcal{Q}$	2
$\tanh(r)$, 1	$T\mathbb{C}P^k \ominus (\mathcal{C} \ominus \mathcal{Q})$	$2k-2$
$\coth(r)$, 1	$\nu\mathbb{C}P^k \ominus \mathbb{C}\nu M$	$2k-2$

(iv) *The shape operator S of M and the structure tensor field ϕ of M commute with each other, that is, $S\phi = \phi S$,*
(v) *The Reeb flow on M is an isometric flow.*

5 The Codazzi equation and some consequences

From the explicit expression of the Riemannian curvature tensor of the complex hyperbolic quadric Q^{m*}, we can easily derive the Codazzi equation for a real hypersurface M in complex hyperbolic quadric Q^{m*} as follows:

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, Z)\\
&= -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y)\\
&\quad - \rho(X)g(BY, Z) + \rho(Y)g(BX, Z)\\
&\quad + \eta(BX)g(BY, \phi Z) + \eta(BX)\rho(Y)\eta(Z)\\
&\quad - \eta(BY)g(BX, \phi Z) - \eta(BY)\rho(X)\eta(Z)
\end{aligned}$$

for any vector fields X, Y and Z tangent to M in Q^{m*}. We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^{m*} satisfies

$$S\xi = \alpha\xi$$

with the Reeb function $\alpha = g(S\xi, \xi)$ on M. Inserting $Z = \xi$ into the Codazzi equation leads to

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = 2g(\phi X, Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX).$$

On the other hand, we have

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X)\\
&= d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y).
\end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$d\alpha(Y) = d\alpha(\xi)\eta(Y) + 2\delta\rho(Y),$$

where the function $\delta = g(AN, N)$ is defined in Sect. 3. Reinserting this into the previous equation yields

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&= -2\delta\eta(X)\rho(Y) + 2\delta\rho(X)\eta(Y)\\
&\quad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y).
\end{aligned}$$

Altogether this implies

$$
\begin{aligned}
0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\
&\quad - 2\delta\rho(X)\eta(Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX) + 2\delta\eta(X)\rho(Y) \\
&= g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y) \\
&\quad - 2\rho(X)\eta(BY + \delta Y) + 2\rho(Y)\eta(BX + \delta X) \\
&= g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y) \\
&\quad - 2\rho(X)g(Y, B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\rho(Y).
\end{aligned}
$$

If $AN = N$ we have $\rho = 0$, otherwise we can use Lemma 3.2 to calculate $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$. Thus we have proved

Lemma 5.1 *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then we have*

$$
(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\phi B\xi.
$$

If the unit normal vector field N is $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ such that $AN = N$. Then we have $\rho = 0$ and $\phi B\xi = -\phi\xi = 0$, and therefore,

$$
2S\phi S - \alpha(\phi S + S\phi) = -2\phi.
$$

If N is not $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ as in Lemma 3.2 and get

$$
\begin{aligned}
&\rho(X)(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi B\xi \\
&= -g(X, \phi(B\xi + \delta\xi))(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi(B\xi + \delta\xi) \\
&= ||B\xi + \delta\xi||^2(g(X, U)\phi U - g(X, \phi U)U) \\
&= \sin^2(2t)(g(X, U)\phi U - g(X, \phi U)U),
\end{aligned}
$$

which is equal to 0 on $\mathcal{Q}$ and equal to $\sin^2(2t)\phi X$ on $\mathcal{C} \ominus \mathcal{Q}$. Altogether we have proved:

Lemma 5.2 *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then the tensor field*

$$
2S\phi S - \alpha(\phi S + S\phi)
$$

leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have

$$
2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}
$$

and

$$
2S\phi S - \alpha(\phi S + S\phi) = -2\delta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},
$$

where $\delta = \cos 2t$ as in Sect. 3.

Now let us assume that M is a real hypersurface in Q^m with isometric Reeb flow. Then the commuting shape operator $S\phi = \phi S$ implies $S\xi = \alpha\xi$, that is, M is *Hopf*. We will now prove that the Reeb curvature α of a Hopf hypersurface is constant if the normal vectors are $\mathfrak{A}$-isotropic. Assume that the unit normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then we have $\delta = 0$ and we get

$$
Y\alpha = d\alpha(\xi)\eta(Y)
$$

for all $Y \in TM$. Since $\mathrm{grad}^M \alpha = d\alpha(\xi)\xi$, we can compute the Hessian $\mathrm{hess}^M \alpha$ by

$$\begin{aligned}(\mathrm{hess}^M \alpha)(X, Y) &= g(\nabla_X \mathrm{grad}^M \alpha, Y)\\ &= d(d\alpha(\xi))(X)\eta(Y) + d\alpha(\xi)g(\phi SX, Y).\end{aligned}$$

As $\mathrm{hess}^M \alpha$ is a symmetric bilinear form, the previous equation implies

$$d\alpha(\xi)g((S\phi + \phi S)X, Y) = 0$$

for all vector fields X, Y on M which are tangential to $\mathcal{C}$.

Now let us assume that $S\phi + \phi S = 0$. For every principal curvature vector, $X \in \mathcal{C}$ such that $SX = \lambda X$ this implies $S\phi X = -\phi SX = -\lambda\phi X$. We assume $||X|| = 1$ and put $Y = \phi X$. Using the normal vector field, N is $\mathfrak{A}$-isotropic, that is $\delta = 0$ in Lemma 5.1, we know that

$$-\lambda^2\phi X + \phi X = \rho(X)B\xi + g(X, B\xi)\phi B\xi.$$

From this, taking the inner product with ϕX and using $g(X, B\xi) = g(X, A\xi) = -g(\phi X, AN) = -\rho(\phi X)$, we have

$$\begin{aligned}-\lambda^2 + 1 &= \rho(X)\eta(B\phi X) - \rho(\phi X)\eta(BX)\\ &= g(X, AN)^2 + g(X, A\xi)^2 = ||X_{\mathcal{C}\ominus\mathcal{Q}}||^2 \leq 1,\end{aligned}$$

where $X_{\mathcal{C}\ominus\mathcal{Q}}$ denotes the orthogonal projection of X onto $\mathcal{C} \ominus \mathcal{Q}$.

On the other hand, from the commuting shape operator and the above equation for $SX = \lambda X$, it follows that

$$-\lambda\phi X = -\phi SX = S\phi X = \phi SX = \lambda\phi X.$$

This gives that the principal curvature $\lambda = 0$. Then the above two equation give $||X_{\mathcal{C}\ominus\mathcal{Q}}||^2 = 1$ for all principal curvature vectors $X \in \mathcal{C}$ with $||X|| = 1$. This is only possible if $\mathcal{C} = \mathcal{C} \ominus \mathcal{Q}$, or equivalently, if $\mathcal{Q} = 0$. Since $m \geq 3$ this is not possible. Hence, we must have $S\phi + \phi S \neq 0$ everywhere, and therefore, $d\alpha(\xi) = 0$, which implies $\mathrm{grad}^M \alpha = 0$. Since M is connected this implies that α is constant. Thus we have proved:

Lemma 5.3 *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with isometric Reeb flow and $\mathfrak{A}$-isotropic normal vector field N everywhere. Then α is constant.*

6 Proof of Theorem 1.1 and Corollary 1.2

Now let us denote by S the shape operator of a real hypersurface M in the complex hyperbolic quadric Q^{m*}. If a real hypersurface M in Q^{m*} has the shape operator of Codazzi type, that is, $(\nabla_X S)Y = (\nabla_Y S)X$ for any X and Y on M, then by the equation of Codazzi we have

$$\begin{aligned}0 = &-\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y)\\ &- g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z)\\ &- g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z).\end{aligned} \tag{6.1}$$

From this, putting $X = \xi$, we know that

$$\begin{aligned}0 = &-g(\phi Y, Z) - g(\xi, AN)g(AY, Z) + g(Y, AN)g(A\xi, Z)\\ &- g(\xi, A\xi)g(JAY, Z) + g(Y, A\xi)g(JA\xi, Z).\end{aligned} \tag{6.2}$$

Here, let us put $Z=\xi$, then we have

$$\begin{aligned}0=&-g(\xi,AN)g(AY,\xi)+g(Y,AN)g(A\xi,\xi)\\&-g(\xi,A\xi)g(JAY,\xi)+g(Y,A\xi)g(JA\xi,\xi)\\=&-2\Big\{g(\xi,AN)g(AY,\xi)-g(A\xi,\xi)g(Y,AN)\Big\}.\end{aligned}$$

Since $g(A\xi,N)=0$, it follows that

$$g(A\xi,\xi)g(AN,Y)=g(AJN,JN)g(AN,Y)=0.$$

This gives that $\cos 2t=0$ or $g(AN,Y)=0$ for any tangent vector field Y on M. Then it follows that either

$$AN=N \quad \text{or} \quad t=\frac{\pi}{4}.$$

From this, we assert the following lemma.

Lemma 6.1 *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with shape operator of Codazzi type. Then the unit normal vector field N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Then let us consider the first case as follows:
Case (1) N: $\mathfrak{A}$-principal, that is, $AN=N$.
Eq. (6.2) gives the following

$$\begin{aligned}0=&-g(\phi Y,Z)-g(\xi,A\xi)g(JAY,Z)+g(A\xi,Y)g(JA\xi,Z)\\=&-g(\phi Y,Z)+g(JAY,Z),\end{aligned}\tag{6.3}$$

where in the second equality we have used that $g(\xi,A\xi)=g(JN,AJN)=-g(JN,JAN)$ $=-g(JN,JN)=-1$ and $g(JA\xi,Z)=-g(JAJN,Z)=-g(AN,Z)=-g(N,Z)=0$ for any vector fields Y and Z on M in Q^{m*}. Thus we know $g(\phi Y,Z)=g(JAY,Z)$. Then the left-hand side is skew-symmetric, but by the anti-commuting property of $AJ=-JA$, the right-hand side becomes

$$g(JAY,Z)=-g(AY,JZ)=g(Y,JAZ),$$

that is, JA becomes symmetric. This gives us a contradiction. So we conclude that there do not exist any real hypersurfaces in the complex hyperbolic quadric Q^{m*} with parallel shape operator for $\mathfrak{A}$-principal unit normal vector field.

We consider the next case as follows:
Case (2) N: $\mathfrak{A}$-isotropic.
In this case the unit normal vecor field N can be written as $N=\frac{1}{\sqrt{2}}(Z_1+JZ_2)$ for $Z_1,Z_2\in V(A)$. Then it follows that

$$AN=\frac{1}{\sqrt{2}}(Z_1-JZ_2),\ AJN=-\frac{1}{\sqrt{2}}(JZ_1+Z_2),\quad \text{and}\quad JN=\frac{1}{\sqrt{2}}(JZ_1-Z_2).$$

So it gives

$$g(\xi,A\xi)=g(JN,AJN)=0,\ g(\xi,AN)=0,\quad \text{and}\quad g(AN,N)=0.\tag{6.4}$$

From this, we know that AN is a tangent vector. Then by putting $X = AN$ into (6.1), we have

$$\begin{aligned}0 &= \eta(Y)g(\phi AN, Z) + 2\eta(Z)g(\phi AN, Y) - g(AY, Z)\\&\quad + g(Y, A\xi)g(JA^2N, Z)\\&= \eta(Y)g(\phi AN, Z) + 2\eta(Z)g(A\xi, Y) - g(AY, Z) - \eta(Z)g(Y, A\xi)\\&= \eta(Y)g(A\xi, Z) + \eta(Z)g(A\xi, Y) - g(AY, Z),\end{aligned} \tag{6.5}$$

where in the third equality we have used

$$g(\phi AN, Z) = g(JAN, Z) = -g(AJN, Z) = g(A\xi, Z).$$

Then the Eq. (6.5) means that

$$g(AY, Z) = 0$$

for any $Y, Z \in \mathfrak{H}$, where $\mathfrak{H}$ denotes the complex subbundle of TM orthogonal to the Reeb vector field ξ. From this, together with (6.4), the complex conjugation on the complex quadric Q^{m*} can be expressed by

$$A = \begin{bmatrix} 0 & 0 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ * & * & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \end{bmatrix} \tag{6.6}$$

But we know that the complex conjugation is involutive, that is, $A^2 = I$. So the expression (6.6) gives us a contradiction. Accordingly, for $\mathfrak{A}$-isotropic normal vector field N, there do not exit any hypersurfaces in the complex hyperbolic quadric Q^{m*} with shape operator of Codazzi type.

Summing up these two cases, we conclude that there do not exist any real hypersurfaces in the complex hyperbolic quadric Q^{m*} with shape operator of Codazzi type. This completes the proof of our Theorem 1.1. Naturally, if the shape operator is parallel, it is of Codazzi type. Accordingly, as a corollary of Theorem 1.1, we get Corollary 1.2.

7 Proof of Theorem 1.3

Before going to prove Theorem 1.3, first let us see if the shape operator of the tube of radius r over a complex hyperbolic space $\mathbb{C}H^k$ in the complex hyperbolic quadric Q^{2k*} is Reeb parallel or not. In order to do this, let us mention that the shape operator S of the tube commutes with the structure tensor ϕ, that is, $S\phi = \phi S$ as in Proposition 4.1. Then, by using the same method as in Berndt and Suh (see [1], p. 1350050-14), it can be easily verified that the expression of the covariant derivative for the shape operator of M in the complex hyperbolic quadric Q^{m*} becomes

$$\begin{aligned}(\nabla_X S)Y = \{&d\alpha(X)\eta(Y) + g((\alpha S\phi - S^2\phi)X, Y) - \delta\eta(Y)\rho(X)\\&- \delta g(BX, \phi Y) - \eta(BX)\rho(Y)\}\xi\\&- \{\eta(Y)\rho(X) + g(BX, \phi Y)\}B\xi - g(BX, Y)\phi B\xi\\&+ \rho(Y)BX + \eta(Y)\phi X + \eta(BY)\phi BX,\end{aligned}$$

where we have put

$$AY = BY + \rho(Y)N, \quad \rho(Y) = g(AY, N)$$

for a complex conjugation $A \in \mathfrak{A}$. Putting $X = \xi$ and using that the Reeb function α is constant and $\rho(\xi) = 0$ for the $\mathfrak{A}$-isotropic unit normal vector field N of M in the complex hyperbolic quadric Q^{2k*}, we have

$$\begin{aligned}(\nabla_\xi S)Y &= -\{\delta g(B\xi, \phi Y) + \eta(B\xi)\rho(Y)\}\xi \\ &\quad - \{\eta(Y)\rho(\xi) + g(B\xi, \phi Y)\}B\xi - g(B\xi, Y)\phi B\xi \\ &\quad + \rho(Y)B\xi + \eta(BY)\phi B\xi \\ &= -\{g(B\xi, \phi Y) - \rho(Y)\}B\xi \\ &= \{g(\phi B\xi, Y) - g(Y, \phi B\xi\}B\xi \\ &= 0,\end{aligned}$$

where in the third equality we have used

$$\begin{aligned}\rho(Y) &= g(AY, N) = g(Y, AN) \\ &= g(Y, AJ\xi) \\ &= -g(Y, JA\xi) = -g(Y, JB\xi) \\ &= -g(Y, \phi B\xi).\end{aligned}$$

So we conclude that a real hypersurface M in Q^{2k*} with commuting shape operator, that is, $S\phi = \phi S$, has parallel shape operator along the Reeb direction, $\nabla_\xi S = 0$.

Now let us prove our Theorem 1.3 in the introduction. Let us assume $\nabla_\xi S = 0$. Then by putting $X = \xi$ in the equation of Codazzi, we have

$$\begin{aligned}-g((\nabla_Y S)\xi, Z) &= -g(\phi Y, Z) - g(\xi, AN)g(AY, Z) + g(Y, AN)g(A\xi, Z) \\ &\quad - g(\xi, A\xi)g(JAY, Z) + g(Y, A\xi)g(JA\xi, Z).\end{aligned}$$

By the assumption of Theorem 1.3, we know that M is Hopf. Then it follows that

$$\begin{aligned}(\nabla_Y S)\xi &= \nabla_Y(S\xi) - S(\nabla_Y \xi) \\ &= \nabla_Y(\alpha\xi) - S\nabla_Y \xi \\ &= (Y\alpha)\xi + \alpha\phi SY - S\phi SY.\end{aligned}$$

From this, together with the above equation, it follows that

$$\begin{aligned}0 &= \eta(Z)Y\alpha + \alpha g(\phi SY, Z) - g(S\phi SY, Z) \\ &\quad - g(\phi Y, Z) - g(\xi, AN)g(AY, Z) + g(Y, AN)g(A\xi, Z) \\ &\quad - g(\xi, A\xi)g(JAY, Z) + g(Y, A\xi)g(JA\xi, Z).\end{aligned} \tag{7.1}$$

From this, putting $Z = \xi$ and using M is Hopf and $g(A\xi, N) = 0$ in Sect. 5, we have

$$\begin{aligned}0 &= Y\alpha - g(\xi, AN)g(AY, \xi) + g(Y.ZN)g(A\xi, \xi) \\ &\quad - g(\xi, A\xi)g(JAY, \xi) + g(Y, A\xi)g(JA\xi, \xi) \\ &= Y\alpha + 2g(Y, AN)g(\xi, A\xi),\end{aligned} \tag{7.2}$$

where we have used that $g(A\xi, N) = 0$ in Sect. 4. So from (7.2), we know that the Reeb function $\alpha = g(S\xi, \xi)$ for the shape operator of M in Q^{m*} is constant if and only if the unit normal vector field N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic, because $AN = N$ or $g(A\xi, \xi) = 0$ for a complex conjugation $A \in \mathfrak{A}$. Now we summarize it as follows:

Lemma 7.1 *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} with Reeb parallel shape operator. Then the Reeb curvature function $\alpha = g(S\xi, \xi)$ is constant if and only if the unit normal vector field N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

When the unit normal vector field N of M in the complex hyperbolic quadric Q^{m*} is $\mathfrak{A}$-principal and the shape operator is Reeb parallel, by using $AN = N$ we know

$$g(\xi, A\xi) = g(JN, AJN) = -g(JN, JN) = -1.$$

So the Eq. (7.1) becomes

$$\alpha g(\phi SY, Z) - g(S\phi SY, Z) - g(\phi Y, Z) + g(JAY, Z) = 0.$$

This formula can be written as follows:

$$\begin{aligned} 0 &= \alpha g(\phi SZ, Y) - g(S\phi SZ, Y) - g(\phi Z, Y) + g(JAZ, Y) \\ &= -\alpha g(S\phi Y, Z) + g(S\phi SY, Z) + g(\phi Y, Z) + g(JAY, Z). \end{aligned}$$

Then taking sum and subtracting from the above two equations give the following, respectively:

$$\alpha g((\phi S - S\phi)Y, Z) = -2g(JAY, Z) \tag{7.3}$$

and

$$\alpha g((\phi S + S\phi)Y, Z) - 2g(S\phi SY, Z) - 2g(\phi Y, Z) = 0. \tag{7.4}$$

Now first we want to prove the following proposition.

Proposition 7.2 *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal normal vector field and Reeb parallel shape operator. Then M is locally congruent to one of the following*

(1) a tube around the totally geodesic Hermitian symmetric space Q^{m-1*} embedded in Q^{m*},
(2) a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*},
(3) a tube around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m*} as a real space form in Q^{m*},
or otherwise
(4) M has two distinct constant prinicipal curvatures given by

$$\alpha, \quad \lambda = \frac{\alpha^2 - 2}{\alpha}$$

with multiplicities m and $m-1$, respectively.

Proof Before going to give our proof, let us mention the following formulas:

$$\begin{aligned} JAY &= J(BY + \rho(Y)N) \\ &= \phi BY + \eta(BY)N + \rho(Y)JN \\ &= \phi BY - \rho(Y)\xi + \eta(BY)N, \\ g(JAY, Z) &= g(\phi BY - \rho(Y)\xi, Z) = g(\phi BY, Z) - \rho(Y)\eta(Z), \end{aligned}$$

and

$$g(AY, Z) = g(BY + \rho(Y)N, Z) = g(BY, Z).$$

So the Codazzi equation becomes

$$\begin{aligned}(\nabla_X S)Y - (\nabla_Y S)X = &- \eta(X)\phi Y + \eta(Y)\phi X + 2g(\phi X, Y)\xi \\ &- g(X, AN)BY + g(Y, AN)BX \\ &- g(X, A\xi)\{\phi BY - \rho(Y)\xi\} \\ &+ g(Y, A\xi)\{\phi BX - \rho(X)\xi\}.\end{aligned} \tag{7.5}$$

From this, putting $X = \xi$, and using that the shape operator is Reeb parallel, we have the following for any $\mathfrak{A}$-principal unit normal N

$$\begin{aligned}S\phi SY - (Y\alpha)\xi - \alpha\phi SY = &- (\nabla_Y S)\xi \\ = &- \phi Y - g(\xi, A\xi)\{\phi BY - \rho(Y)\xi\} + g(Y, A\xi)\phi B\xi,\end{aligned} \tag{7.6}$$

where we have put $A\xi = B\xi$ and $AX = BX + \rho(X)N$. From this, taking the inner product with ξ, we have

$$Y\alpha = -\rho(Y) = -g(AY, N) = -g(Y, AN) = -g(Y, N) = 0.$$

From this, together with (7.4), (7.6) and $\phi B\xi = 0$ for N is $\mathfrak{A}$-principal, we have

$$\frac{\alpha}{2}(S\phi - \phi S)Y = \phi BY. \tag{7.7}$$

By Lemma 5.1, we know that for the $\mathfrak{A}$-principal unit normal N

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi.$$

Now let us put $SX = \lambda X$ for some $X \in \mathfrak{H}$. Then it follows that

$$(2\lambda - \alpha)S\phi X = (\alpha\lambda - 2)\phi X.$$

When $2\lambda - \alpha = 0$, it gives $\lambda = 1$ and $\alpha = 2$. So it becomes a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*}.

When $2\lambda - \alpha \neq 0$, then

$$S\phi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X. \tag{7.8}$$

In this case, $Y \in T_z Q^{m*} = V(A) \oplus JV(A)$. So we consider the following three cases.

Subcase 1. $BY = Y$ for $Y \in V(A)$.

Then by (7.7) and (7.8), we have

$$\frac{\alpha}{2}\left\{\frac{\alpha\lambda - 2}{2\lambda - \alpha} - \lambda\right\}\phi Y = \phi Y.$$

This gives that the principal curvatures satisfy $\lambda\{\alpha\lambda + (2 - \alpha^2)\} = 0$, which means $\lambda = 0$ or $\lambda = \frac{\alpha^2 - 2}{\alpha}$. The expression of the shape operator S becomes

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \frac{2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

This means equivalently that the shape operator satisfies $S\phi + \phi S = k\phi$, where $k = \frac{2}{\alpha}$. (See Blair [3]). Then by Theorem C in the introduction (see Berndt and Suh [2]), M is a tube of radius r around a totally geodesic Hermitian symmetric space Q^{m-1*} embedded in Q^{m*}, a horosphere in Q^{m*} whose center at infinity is the equivalence class of an $\mathfrak{A}$-principal geodesic in Q^{m*}, or the tube of radius $r \to \infty$ (with infinite radius) around the m-dimensional real hyperbolic space $\mathbb{R}H^m$, which is embedded in Q^{m*} as a real space form, or otherwise the expression of the shape operator becomes

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{\alpha^2-2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \frac{\alpha^2-2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \alpha \end{bmatrix}$$

Subcase 2. $BY = -Y$ for $Y \in V(A)$.

Then by (7.7) and (7.8), we have

$$\frac{\alpha}{2}\left\{\frac{\alpha\lambda - 2}{2\lambda - \alpha} - \lambda\right\}\phi Y = -\phi Y.$$

This gives that the principal curvatures satisfy $\alpha\{\alpha\lambda - 1 - \lambda^2\}\phi Y = -(2\lambda - \alpha)\phi Y$, which implies $(\alpha\lambda - 2)(\lambda - \alpha) = 0$. Then it follows that $\lambda = \alpha$ or $\lambda = \frac{2}{\alpha}$. Then the expressions of the shape operator are the same as given in Subcase 1.

Subcase 3. $Y = \frac{1}{\sqrt{2}}(Z + W)$ for $Z \in V(A)$ and $W \in JV(A)$.

In this subcase, we have $BY = AY = \frac{1}{\sqrt{2}}(Y - Z)$. Then also by (7.7) and (7.8), we have

$$\frac{\alpha}{2}\left\{\frac{\alpha\lambda - 2}{2\lambda - \lambda} - \lambda\right\}\left\{\frac{1}{\sqrt{2}}\phi Y + \frac{1}{\sqrt{2}}\phi Z\right\} = \frac{1}{\sqrt{2}}(\phi Y - \phi Z).$$

Then by comparing ϕZ and ϕW, we have both

$$\frac{\alpha}{2}\left\{\frac{\alpha\lambda - 2}{2\lambda - \lambda} - \lambda\right\} = 1$$

and

$$\frac{\alpha}{2}\left\{\frac{\alpha\lambda - 2}{2\lambda - \lambda} - \lambda\right\} = -1.$$

This gives a contradiction. So this case cannot appear.

Summing up above discussions, we have a complete proof of the above proposition.

Then by virtue of Lemma 7.1 and Proposition 7.2, we are now considering only the case that N is $\mathfrak{A}$-isotropic for M in the complex hyperbolic quadric Q^m. Naturally we can assert the following

Proposition 7.3 *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with non-vanishing Reeb curvature. If the unit normal N is $\mathfrak{A}$-isotropic and the shape operator is Reeb parallel, then M is locally congruent to a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

Proof By Lemma 5.3, we know that the Reeb curvature α is constant, because N is $\mathfrak{A}$-isotropic. Moreover, the unit normal N can be written as $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$, where $Z_1, Z_2 \in V(A)$. Accordingly, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0, g(\xi, AN) = 0, g(AN, N) = 0,$$

because $AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2)$, $AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2)$, and $JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2)$.

From this, using $\nabla_\xi S = 0$ in the equation of Codazzi, we have

$$\begin{aligned} \alpha g(\phi SX, Z) - g(S\phi SX, Z) = g(\phi X, Z) &- g(X, AN)g(A\xi, Z) \\ &- g(X, A\xi)g(JA\xi, Z). \end{aligned} \tag{7.9}$$

On the distribution $\mathcal{Q}$, we know that $AX \in T_z M$, $z \in M$ for any $A \in \mathfrak{A}$. So it follows that $g(X, AN) = g(AX, N) = 0$ and

$$g(JA\xi, Z) = -g(JAJN, Z) = -g(AN, Z) = -g(N, AZ) = 0.$$

On the other hand, by Lemma 5.2 in Sect. 5 due to Berndt and Suh [1], we can use the following formula

$$S\phi S = \frac{\alpha}{2}(S\phi + \phi S) - \phi \tag{7.10}$$

on the distribution $\mathcal{Q}$ in M. From this, together with (7.7), it follows that

$$-\frac{\alpha}{2} g((S\phi - \phi S)X, Z) = 0$$

for any X and Z tangent to M in Q^{m*}. So from the assumption we have that the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$ on the distribution $\mathcal{Q}$. Then together with (7.8), on the distribution $\mathcal{Q}$ we get the following

$$S\phi S - \alpha S\phi = -\phi.$$

When we consider a principal curvature vector $X \in \mathcal{Q}$ such that $SX = \lambda X$, then the principal curvature λ becomes a solution of $x^2 - \alpha x + 1 = 0$. Moreover, this equation has two distinct roots, and we may put $\lambda = \coth r$, $\mu = \tanh r$ and $\alpha = 2 \coth 2r$.

Now let us continue our discussion on the distribution $\mathcal{C} \ominus \mathcal{Q}$. Then by Lemma 5.2 in Sect. 5, we know that

$$2S\phi S - \alpha(S\phi + \phi S) = 0 \tag{7.11}$$

because $\delta = 0$ for an $\mathfrak{A}$-isotropic normal vector field N. Now let us differentiate $g(\xi, AN) = 0$. Then it follows that

$$g(\bar{\nabla}_X \xi, AN) + g(\xi, (\bar{\nabla}_X A)N + A\bar{\nabla}_X N) = 0.$$

From this, together with $(\bar{\nabla}_X A)N = q(X)AN$, we have

$$\begin{aligned} 0 &= g(\phi SX, AN) - g(\xi, ASX) \\ &= g(\phi SX, AN) + g(JN, ASX) \\ &= g(\phi SX, AN) + g(N, A\phi SX) + \eta(SX)g(N, A\xi) \\ &= -2g(S\phi AN, X) \end{aligned} \tag{7.12}$$

for any vector field X on the distribution $\mathcal{C} \ominus \mathcal{Q}$. So $S\phi AN = 0$ is equivalent to $SA\xi = 0$. From this, together with (7.11), we have

$$\alpha S\phi A\xi = 0.$$

So we get $S\phi A\xi = 0$ from the assumption. This means that $S\phi = \phi S$ on the distribution $\mathcal{C}\ominus\mathcal{Q} = \text{Span}\{A\xi, AN\}$, where $AN = -\phi A\xi$. Consequently, we conclude that the shape operator S commutes with the structure tensor ϕ for a Hopf hypersurface M in Q^{m*}. This means that the Reeb flow of M is isometric. Then by Theorem A, we give a complete proof of our proposition.

Summing up the above discussions with Lemma 7.1, Propositions 7.2 and 7.3, we give a complete proof of Theorem 1.3 in the introduction.

Acknowledgements The present authors would like to express their deep gratitude to the referee for his/her valuable comments to develop the first version of this manuscript.

References

1. Berndt, J., Suh, Y.J.: Real hypersurfaces with isometric Reeb flow in complex quadrics. Int. J. Math. **24**, 1350050 (2013)
2. Berndt, J., Suh, Y.J.: Contact hypersurfaces in Kaehler manifold. Proc. AMS **143**, 2637–2649 (2015)
3. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics. Springer, Berlin (2010)
4. Dragomir, S., Perrone, D.: Harmonic Vector Fields: Variational Principles and Differential Geometry. Elsevier, Amsterdam (2011)
5. Klein, S.: Totally geodesic submanifolds in the complex quadric. Differ. Geom. Appl. **26**, 79–96 (2008)
6. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II. A Wiley-Interscience Publ., Wiley Classics Library Ed., New York (1996)
7. Montiel, S., Romero, A.: On some real hypersurfaces of a complex hyperbolic space. Geom. Dedic. **20**, 245–261 (1986)
8. Okumura, M.: On some real hypersurfaces of a complex projective space. Trans. Am. Math. Soc. **212**, 355–364 (1975)
9. Pérez, J.D.: Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} A = 0$. J. Geom. **49**, 166–177 (1994)
10. Pérez, J.D., Suh, Y.J.: Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$. Differ. Geom. Appl. **7**, 211–217 (1997)
11. Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel and commuting Ricci tensor. J. Korean Math. Soc. **44**, 211–235 (2007)
12. Pérez, J.D., Suh, Y.J., Watanabe, Y.: Generalized Einstein real hypersurfaces in complex two-plane Grassmannians. J. Geom. Phys. **60**, 1806–1818 (2010)
13. Reckziegel, H.: On the geometry of the complex quadric. In: Franki, D., Boris, K., Udo, S., Ignace, Van de Woestyne., Leopold, V. (eds.) Geometry and Topology of Submanifolds VIII (Brussels/Nordfjordeid 1995), pp. 302–315. World Sci. Publ., River Edge (1995)
14. Smyth, B.: Differential geometry of complex hypersurfaces. Ann. Math. **85**, 246–266 (1967)
15. Suh, Y.J.: Real hypersurfaces of type B in complex two-plane Grassmannians. Monatsh. Math. **147**, 337–355 (2006)
16. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc. R. Soc. Edinb. A **142**, 1309–1324 (2012)
17. Suh, Y.J.: Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv. Appl. Math. **50**, 645–659 (2013)
18. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature. J. Math. Pures Appl. **100**, 16–33 (2013)
19. Suh, Y.J.: Real hypersurfaces in the complex quadric with Reeb parallel shape operator. Int. J. Math. **25**, 1450059 (2014)
20. Suh, Y.J.: Real hypersurfaces in the complex quadric with parallel Ricci tensor. Adv. Math. **281**, 886–905 (2015)
21. Suh, Y.J.: Real hypersurfaces in complex quadric with harmonic curvature. J. Math. Pures Appl. **106**, 393–410 (2016)
22. Suh, Y.J.: Isometric Reeb flow in complex hyperbolic quadric. arXiv:1608.02290 [math.DG] (2016)